J. Math. Anal. Appl. 354 (2009) 24–30

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Estimates for the lowest eigenvalue of a star graph

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ARTICLE INFO

Article history:
Received 6 November 2008
Available online 11 December 2008
Submitted by Goong Chen

Keywords:
Quantum wires
Spectral theory
Eigenvalue

ABSTRACT

We derive new estimates for the lowest eigenvalue of the Schrödinger operator associated with a star graph in $\mathbf{R}^2$. We achieve this by a variational method and a procedure for identifying test functions which are sympathetic to the geometry of the star graph.

1. Introduction

A star graph Γ_N in the plane $\mathbf{R}^2$ is the union of a finite number N of rays $\theta = \theta_i$ $(1 \leqslant i \leqslant N)$ emanating from the origin O of polar coordinates (r, θ). The concept of a star graph was featured by Exner and Němcová [5,6], in the context of the spectrum σ associated with the perturbed Laplacian

$$H := -\Delta - k\delta(\mathbf{x} - \Gamma_N) \tag{1.1}$$

in $\mathbf{R}^2$, where δ is the Dirac delta function, $\mathbf{x} = (r, \theta)$ and k (> 0) is a given real constant. The geometry of Γ_N influences the nature of σ, and we refer to [6] and to the survey paper [3] for what is known, not only in the case of Γ_N but also for more general graphs Γ.

The particular properties of σ which are our concern in this paper are

(i) the essential spectrum σ_{ess} is the interval $[-k^2/4, \infty)$,
(ii) the discrete spectrum σ_d is non-empty except when $N = 2$ and Γ_2 is a single straight line

[6, Proposition 5.4 and Theorem 5.7]; see also [2].

In a recent paper [1], we observed that the proof of (ii) in [6, Theorem 5.7] is technically complicated, relying as it does on a more general theory of Exner and Ichinose [4]. Our aim in [1] was to give a simple proof of (ii) by identifying a real-valued function $f \in W^{1,2}(\mathbf{R}^2)$ for which the variational quotient

$$V(f) := \left(\int_{\mathbf{R}^2} |\nabla f|^2 \, d\mathbf{x} - k \int_{\Gamma_N} f^2 \, dr \right) \Big/ \int_{\mathbf{R}^2} f^2 \, d\mathbf{x} \tag{1.2}$$

satisfies

$$V(f) < -k^2/4. \tag{1.3}$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.12.014

When $N \geqslant 4$, the choice of f is very simple: $f(\mathbf{x}) = \exp(-kNr/2\pi)$ gives

$$V(f) = -k^2N^2/4\pi^2 < -k^2/4 \tag{1.4}$$

[1, Section 2]. When $N = 2$, a less simple f produces (1.3) provided that the two rays contain an angle not exceeding 0.9271 ($= 53.12^\circ$) and there is a similar partial result when $N = 3$ [1, Sections 3 and 4].

In this paper, we adopt a systematic method of choosing f which is more sympathetic to the geometry of Γ_N and which not only improves on (1.4) but also covers the case $N = 3$ in full. Noting that the lowest eigenvalue $\lambda_{0,N}$ in σ_d satisfies the variational inequality $\lambda_{0,N} \leqslant V(f)$ (cf. [6, Section 5.2]), we can state one of our main results as

$$\lambda_{0,N} \leqslant -2k^2N^2\bigl(N^2+\pi^2\bigr)^2/\bigl\{\pi^2\bigl(2N^2+\pi^2\bigr)\bigl(4N^2+5\pi^2\bigr)\bigr\}. \tag{1.5}$$

In particular, when $N = 3$ this gives

$$\lambda_{0,3} \leqslant -(0.273)k^2 < -k^2/4$$

for all Γ_3. We do not, however, have anything to add here to [1] concerning $N = 2$.

In Section 2, we propound two choices for f and explain the rationale for these choices. Then, in Section 3, we derive the two corresponding estimates for $\lambda_{0,N}$, of which (1.5) is one. Finally, Section 4 contains a discussion and comparison of these results.

2. Two choices for f

We begin by considering

$$f(\mathbf{x}) = \exp\bigl\{-\arg(\theta)\bigr\} \tag{2.1}$$

in (1.2), where the parameter a and the function g are at our choice. Then, first,

$$\int_{\mathbf{R}^2} |\nabla f|^2\,d\mathbf{x} = a^2\int_0^{2\pi}\int_0^{\infty}\bigl\{g^2(\theta)+g'^2(\theta)\bigr\}\exp\bigl\{-2\arg(\theta)\bigr\}r\,dr\,d\theta = \frac{1}{4}\int_0^{2\pi}\bigl(1+g'^2/g^2\bigr)\,d\theta = \frac{1}{4}I_1, \tag{2.2}$$

say. Similarly,

$$\int_{\mathbf{R}^2} f^2\,d\mathbf{x} = \frac{1}{4a^2}\int_0^{2\pi} 1/g^2\,d\theta = \frac{1}{4a^2}I_2 \tag{2.3}$$

say. Again

$$\int_{\Gamma_N} f^2\,dr = \frac{1}{2a}\sum_1^N 1/g(\theta_i) = s/2a \tag{2.4}$$

say. Then, by (1.2),

$$V(f) = \bigl(a^2I_1 - 2aks\bigr)/I_2.$$

Completing the square in a and choosing $a = ks/I_1$, we obtain an estimate for the lowest eigenvalue in the form

$$\lambda_{0,N} \leqslant V(f) = -k^2s^2/I_1I_2. \tag{2.5}$$

2.1. The first choice for f

Let β and b be parameters such that $0 < \beta < 2\pi$ and $0 < b < 1$, and let t be a variable in $[0,\beta]$. Then we begin by defining a function G of t by

$$G(t;\beta,b) = \left\{b + 2\frac{1-b}{\beta}\left|\frac{1}{2}\beta - t\right|\right\}^{-1/2} \quad (0 \leqslant t \leqslant \beta). \tag{2.6}$$

We note that

$$G(0;\beta,b) = G(\beta;\beta,b) = 1 \tag{2.7}$$

and, after a short calculation,

$$\int_0^\beta G'^2/G^2\,dt = (1-b)^2/\beta b, \tag{2.8}$$

$$\int_0^\beta 1/G^2\,dt = \frac{1}{2}\beta(1+b). \tag{2.9}$$

Now suppose that the rays $\theta = \theta_i$ of Γ_N are labelled in order of increasing θ_i and that $\theta_{N+1} := \theta_1 + 2\pi$. Then the angles between the rays are

$$\beta_i = \theta_{i+1} - \theta_i \quad (1 \leqslant i \leqslant N).$$

We now define the function $g(\theta)$ in (2.1) by

$$g(\theta) = G(\theta - \theta_i; \beta_i, b_i) \quad (\theta_i \leqslant \theta \leqslant \theta_{i+1},\ 1 \leqslant i \leqslant N) \tag{2.10}$$

where the parameters b_i $(0 < b_i < 1)$ are at our disposal. By (2.7), $g(\theta)$ is continuous as θ passes through θ_i. It follows from (2.2)–(2.4) and (2.7)–(2.9) that

$$s = N \tag{2.11}$$

and

$$I_1 = 2\pi + S_1, \qquad I_2 = \frac{1}{2}(2\pi + S_2), \tag{2.12}$$

where

$$S_1 = \sum_1^N \frac{(1-b_i)^2}{\beta_i b_i} \tag{2.13}$$

and

$$S_2 = \sum_1^N \beta_i b_i. \tag{2.14}$$

We leave until the next section the choice of the b_i.

2.2. The second choice of f

This time we start with just one parameter β such that $0 < \beta < \pi$, and we define

$$G(t;\beta) = \left(\sec\frac{1}{2}\beta\right)\cos\left(t - \frac{1}{2}\beta\right) \quad (0 \leqslant t \leqslant \beta). \tag{2.15}$$

Then

$$G(0;\beta) = G(\beta;\beta) = 1$$

and

$$\int_0^\beta \left(1 + G'^2/G^2\right)dt = 2\tan\frac{1}{2}\beta,$$

$$\int_0^\beta 1/G^2\,dt = \sin\beta.$$

Similarly to (2.10), we now define

$$g(\theta) = G(\theta - \theta_i; \beta_i) \quad (\theta_i \leqslant \theta \leqslant \theta_{i+1},\ 1 \leqslant i \leqslant N) \tag{2.16}$$

and we obtain

$$s = N, \qquad I_1 = 2\sum_1^N \tan\frac{1}{2}\beta_i, \qquad I_2 = \sum_1^N \sin\beta_i. \tag{2.17}$$

2.3. Rationale of the choices

The second choice is guided by the fact that, when $N = 4$ and Γ_4 is the symmetric graph with all $\beta_i = \pi/2$, the lowest eigenvalue $\lambda_{0,4}$ and corresponding eigenfunction $\psi_{0,4}(\mathbf{x})$ are known exactly:

$$\lambda_{0,4} = -\frac{1}{2}k^2, \qquad \psi_{0,4}(\mathbf{x}) = \exp\left\{-\frac{1}{2}k\big(|x| + |y|\big)\right\},$$

as is easily verified [6, Example 5.2]. Thus $\psi_{0,4}$ arises from (2.1) with $a = \frac{1}{2}k$ and the choice (2.15) with $\beta = \pi/2$. More generally, the level curves of f with (2.15) consist of straight line segments, as is the case with $\psi_{0,4}$.

The first choice is guided by Fig. 10 in [6], where the level curves are shown for the lowest eigenfunction for the symmetric Γ_6. The feature here is that the value of r on any one level curve is greatest when $\theta = \theta_i$. This feature is reproduced in our first choice (2.6) and (2.10) with, at the same time, a workable evaluation of the integrals in (2.8) and (2.9).

3. Estimates for the lowest eigenvalue

In this section, the two theorems give the estimates for $\lambda_{0,N}$ which follow from the two choices of f in Sections 2.1 and 2.2.

Theorem 3.1. *For any configuration of Γ_N,*

$$\lambda_{0,N} \leqslant -2k^2N^2\big(N^2 + \pi^2\big)^2 \big/ \big\{\pi^2\big(2N^2 + \pi^2\big)\big(4N^2 + 5\pi^2\big)\big\}. \tag{3.1}$$

Proof. In the formulae of Section 2.1, we choose

$$b_i = 1/(1 + p\beta_i),$$

where p (> 0) is a further parameter at our disposal. Then (2.13) and (2.14) become $S_1 = p^2 S$ and $S_2 = S$, where

$$S = \sum_1^N \beta_i/(1 + p\beta_i). \tag{3.2}$$

Then again (2.5), (2.11) and (2.12) give

$$\lambda_{0,N} \leqslant -2k^2N^2 \big/ \big\{\big(2\pi + p^2 S\big)(2\pi + S)\big\}. \tag{3.3}$$

In order to obtain an inequality (3.3) valid for any configuration of Γ_N, we require the maximum value of S considered as a function of the β_i, subject to $\sum \beta_i = 2\pi$. It is easy to check from (3.2) that this maximum occurs when all $\beta_i = 2\pi/N$, that is, when Γ_N is symmetric. Then $S_{\max} = 2N\pi/(N + 2p\pi)$ and (3.3) becomes

$$\lambda_{0,N} \leqslant -\frac{k^2N^2}{2\pi^2} \bigg/ \left\{\left(1 + \frac{p^2N}{N + 2p\pi}\right)\left(1 + \frac{N}{N + 2p\pi}\right)\right\} \tag{3.4}$$

for all Γ_N.

Next, in order to make (3.4) the best it can be, we choose p to minimise the denominator

$$\phi(p) := \left(1 + \frac{p^2N}{N + 2p\pi}\right)\left(1 + \frac{N}{N + 2p\pi}\right).$$

The equation $\phi'(p) = 0$ can be written as a cubic in p:

$$2p(N + p\pi)^2 - \pi\big(p^2N + 2p\pi + N\big) = 0. \tag{3.5}$$

The exact solution (as given by Cardan's formula) is not particularly helpful, but we propose the value

$$p = \pi/2N \tag{3.6}$$

as a good approximation to the exact solution for the following reasons. When N is large, we write $p = Nq$ in (3.5) to obtain $q = \pi/2N^2 + O(N^{-4})$. Thus (3.6) is correct asymptotically as $N \to \infty$. For smaller N, the comparison between the exact solution of (3.5) and (3.6) is (to 2 d.p.) in Table 1.

Thus $p = \pi/2N$ is our choice of p in (3.4), and then (3.1) follows immediately. □

We emphasise that (3.1) covers all Γ_N: for particular configurations, we may expect (3.3) to lead to an improvement of (3.1) and we give some examples of this in Section 4.3.

Table 1

N	exact p	(3.6)
3	0.52	0.52
4	0.41	0.39
5	0.33	0.31
6	0.27	0.26
10	0.16	0.16

Theorem 3.2. *Let $0 < \beta_i < \pi$ $(1 \leqslant i \leqslant N)$. Then*

$$\lambda_{0,N} \leqslant -N^2k^2/(2ST), \tag{3.7}$$

where

$$S = \sum_1^N \sin\beta_i, \qquad T = \sum_1^N \tan\frac{1}{2}\beta_i. \tag{3.8}$$

Proof. This follows immediately from (2.5) and (2.17). □

Corollary 3.3. *Let $\beta = \max\beta_i$ $(1 \leqslant i \leqslant N)$ and suppose that $\beta \leqslant \pi/2$. Then*

$$\lambda_{0,N} \leqslant -\frac{1}{4}k^2\operatorname{cosec}^2\frac{1}{2}\beta. \tag{3.9}$$

Proof. By (3.8), we have $S \leqslant N\sin\beta$, $T \leqslant N\tan\frac{1}{2}\beta$, and hence (3.7) gives

$$\lambda_{0,N} \leqslant -k^2\Big/\left(2\sin\beta\tan\frac{1}{2}\beta\right) = -\frac{1}{4}k^2\operatorname{cosec}^2\frac{1}{2}\beta. \qquad \square$$

We note that the condition $\beta \leqslant \pi/2$ excludes $N = 3$ – but see Corollary 3.5 below.

Corollary 3.4. *Let N be even, $N = 2M$, and suppose that Γ_N has M equal angles β and M equal angles $2\pi/M - \beta$ $(\beta \leqslant \pi/M)$. Then*

$$\lambda_{0,N} \leqslant -\frac{1}{2}k^2\left\{\cos\frac{\pi}{M}\sec\left(\frac{\pi}{M}-\beta\right)+1\right\}\operatorname{cosec}^2\frac{\pi}{M}. \tag{3.10}$$

Proof. By (3.8), we now have

$$S = M\sin\beta + M\sin\left(\frac{2\pi}{M}-\beta\right) = 2M\sin\frac{\pi}{M}\cos\left(\frac{\pi}{M}-\beta\right)$$

and

$$T = M\tan\frac{1}{2}\beta + M\tan\left(\frac{\pi}{M}-\frac{1}{2}\beta\right) = 2M\left(\sin\frac{\pi}{M}\right)\Big/\left\{\cos\frac{\pi}{M}+\cos\left(\frac{\pi}{M}-\beta\right)\right\}.$$

Thus

$$2ST = 2N^2\left(\sin^2\frac{\pi}{M}\right)\Big/\left\{\cos\frac{\pi}{M}\sec\left(\frac{\pi}{M}-\beta\right)+1\right\}$$

and (3.10) follows. □

When $\beta = \pi/M$ in Corollary 3.4, Γ_N is the symmetric graph with all angles $2\pi/N$ and (3.10) becomes

$$\lambda_{0,N} \leqslant -\frac{1}{4}k^2\operatorname{cosec}^2(\pi/N). \tag{3.11}$$

In the next corollary, we note that this particular estimate is true for odd N as well.

Corollary 3.5. *The inequality* (3.11) *holds for all symmetric Γ_N.*

Proof. In (3.8), we have $S = N\sin(2\pi/N)$, $T = N\tan(\pi/N)$. Then (3.11) follows immediately from (3.7), similarly to (3.9). □

Finally in this section, we note that, when $N = 3$, we can write (3.8) as

$$S = 4\prod_{i=1}^{3}\sin\frac{1}{2}\beta_i, \qquad T = \prod_{i=1}^{3}\tan\frac{1}{2}\beta_i$$

by elementary trigonometry [7, Section 127]. Thus (3.7) can be written alternatively as

$$\lambda_{0,3} \leqslant -\frac{9}{8}k^2\prod_{i=1}^{3}\frac{\cos\frac{1}{2}\beta_i}{(\sin\frac{1}{2}\beta_i)^2}. \tag{3.12}$$

4. Discussion and examples

4.1. A maximum conjecture

It is a natural conjecture [3, Section 7.4] that, for a given N, the lowest eigenvalue $\lambda_{0,N}$ is maximised when Γ_N is symmetric. Some evidence (supportive rather than conclusive) is that the right-hand side of (3.3) achieves its greatest value in terms of the β_i when Γ_N is symmetric, as we noted when we used (3.3).

We can however go further and state that the conjecture is correct in the particular case when $N = 4$ and the angles in Γ_4 are β, β, $\pi - \beta$, $\pi - \beta$ (in any order). For this type of Γ_4, we have

$$S = 4\sin\beta, \qquad T = 2\left(\tan\frac{1}{2}\beta + \cot\frac{1}{2}\beta\right)$$

in (3.8), and then (3.7) gives $\lambda_{0,4} \leqslant -\frac{1}{2}k^2$. As we mentioned in Section 2.3, $-\frac{1}{2}k^2$ is precisely the lowest eigenvalue for the symmetric Γ_4.

4.2. Symmetric Γ_6

For the symmetric Γ_6, we have $\beta = \pi/3$ in (3.9), and so $\lambda_{0,6} \leqslant -k^2$ in this case. However, in [6, Fig. 10], it is stated that $\lambda_{0,6} = -(0.612)k^2$. This approximate value of $\lambda_{0,6}$ is obtained in [6] by a cut-off of Γ_6 at $r = 30$ and then using an approximation to (1.1) by 601 point potentials. There is thus a discrepancy concerning the value of $\lambda_{0,6}$ which would no doubt be reduced if a greater cut-off than $r = 30$ were used.

4.3. Comparison of the two choices

Our two choices for f in Section 2 lead to the inequalities (3.3) and (3.7). Here we compare these two results for both symmetric and non-symmetric Γ_N. The figures in Tables 2–5 refer to the multiples of $-k^2$ which arise from (3.3) and (3.7).

4.3.1. Symmetric graphs

Here the relevant inequalities for the two cases are (3.1) and (3.11), the latter being the better of the two. See Table 2.

4.3.2. A non-symmetric example

Here we consider $N = 4$ and the angles in Γ_4 are β, β, β and $2\pi - 3\beta$. For the last angle to exist, we require $\beta < 2\pi/3$ $(= 2.09)$. Also, for (3.7) to be applicable, we require $2\pi - 3\beta < \pi$, that is, $\beta > \pi/3$ $(= 1.05)$.

In (3.2), we now have

$$S = \frac{3\beta}{1 + p\beta} + \frac{2\pi - 3\beta}{1 + p(2\pi - 3\beta)}.$$

For each β, we have to choose p to minimise the denominator in (3.3). We do this computationally for a mesh of values of β and, in Table 3, we give the values of β, the corresponding values of p, and the multiples of $-k^2$ from (3.3) and (3.7). These figures (and intermediate ones not displayed here) show that (3.7) is better than (3.3) when $1.3 \leqslant \beta < 2.09$, a range which includes the value $\pi/2$ for the symmetric Γ_4. Also, (3.3) and (3.7) together confirm the conjecture in Section 4.1 for this type of Γ_4 when $\beta \leqslant 0.6$ $(= 34.38^\circ)$ and when $\beta \geqslant 1.6$ or, more exactly, $\beta \geqslant \pi/2$.

4.3.3. Non-symmetric examples with $N = 6$

Here we first consider similarly, but briefly, the case $N = 6$ and the angles in Γ_6 are β (three times) and $2\pi/3 - \beta$ (three times) in any order, where $\beta \leqslant \pi/3$ without loss of generality. This time (3.2) gives

$$S = 3\left(\frac{\beta}{1 + p\beta} + \frac{2\pi - 3\beta}{3 + p(2\pi - 3\beta)}\right),$$

and the other relevant inequality is (3.10) with $M = 3$. Thus (3.10) is always better than (3.3) in this example. See Table 4.

For our final example, we take the angles in Γ_6 to be β, 2β, $\pi - 3\beta$ (twice each). This time (3.3) is better when $\beta \leqslant 0.09$ $(= 5.16^\circ)$. See Table 5.

Table 2

N	(3.1)	(3.11)
3	0.273	0.333
4	0.457	0.500
5	0.689	0.724
6	0.970	1.000
10	2.594	2.618

Table 3

β	p	(3.3)	(3.7)
0.01	0.805	0.624	N/A
0.1	0.662	0.594	N/A
0.4	0.510	0.533	N/A
0.6	0.470	0.507	N/A
0.7	0.457	0.497	N/A
1.1	0.423	0.469	0.196
1.2	0.418	0.465	0.390
1.3	0.413	0.461	0.467
1.5	0.408	0.458	0.499
1.6	0.407	0.457	0.500
2.0	0.460	0.473	0.553

Table 4

β	p	(3.3)	(3.10)
0.01	0.509	1.088	1.322
0.3	0.365	1.020	1.121
0.6	0.303	0.986	1.036
1.0	0.273	0.970	1.000

Table 5

β	p	(3.3)	(3.7)
0.01	0.637	1.194	1.125
0.09	0.506	1.125	1.121
0.1	0.495	1.118	1.119
1.0	0.372	1.022	1.094

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